MATH20132 Calculus of Several Variables. 2020-21

Problems 1 : Limits

Questions 1 - 4 concern the limits of functions. *The* ε - δ *definition of a limit is that* $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *has the limit* $\mathbf{b} \in \mathbb{R}^m$ *at* $\mathbf{a} \in U$ *iff*

$$\forall \varepsilon > 0, \exists\, \delta > 0 : \forall \mathbf{x} \in \mathbb{R}^n,\ 0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \varepsilon.$$

1. By verifying the ε - δ definition of limit show that the scalar-valued function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y)^T \mapsto x + y$ has limit 5 at $\mathbf{a} = (2, 3)^T$.

Hint At some point in verifying the definition you assume $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$. In particular this gives **two** pieces of information, namely that $|x - 2| < \delta$ and $|y - 3| < \delta$.

2. By verifying the ε - δ definition of limit show that the scalar-valued function $g : \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \mapsto xy + x + y$ has limit 11 at $\mathbf{a} = (2, 3)^T$.

Hint Perhaps start by proving that

$$xy + x + y - 11 = (x - 2)(y - 3) + 4(x - 2) + 3(y - 3).$$

Deduce that if $|x - 2| < \delta$, $|y - 3| < \delta$ and $\delta \leq 1$ then $|xy - 6| < 8\delta$. Now look at the definition of limit.

3. By verifying the ε - δ definition of limit show that the vector-valued function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix},$$

has limit $(7, -7)^T$ at $\mathbf{a} = (2, 3)^T$.

Note For practice I have asked you to verify the definition, **not** to use any result from the course that would allow you to look at each component separately.

4. By verifying the ε - δ definition of limit show that the vector-valued $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ xy + x + y \end{pmatrix},$$

has limit $(5,11)^T$ at $\mathbf{a} = (2,3)^T$.

Hint try to make use of the results used in Questions 1 and 2.

5. Assume $f,\, g : D \subseteq \mathbb{R}^n \to \mathbb{R}$ are **scalar-valued** functions with domain D containing a deleted neighbourhood of $\mathbf{a} \in \mathbb{R}^n$. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = b \in \mathbb{R}$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = c \in \mathbb{R}$ prove that

i. $\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x}) + g(\mathbf{x})) = b + c$,

ii. $\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x})\, g(\mathbf{x})) = bc$ and

iii. $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) / g(\mathbf{x}) = b/c$ provided $c \neq 0$.

Hint No new ideas are required, the proofs are identical to those for functions of one variable.

The following is a corollary of Question 5.

6. Assume $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are **vector-valued** functions with domain D containing a deleted neighbourhood of $\mathbf{a} \in \mathbb{R}^n$. If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \in \mathbb{R}^m$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{c} \in \mathbb{R}^m$ prove that

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \bullet \mathbf{g}(\mathbf{x}) = \mathbf{b} \bullet \mathbf{c}.$$

Here $\bullet$ is the scalar or dot product of vectors.

Hint Make use of the previous question.

7. **Lemma from Lecture Notes, limits along straight lines** *Assume* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is a vector-valued function with domain* A *containing a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$. *Assume* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$. *Then, for any non-zero vector* $\mathbf{v} \in \mathbb{R}^n$*, the directional limit of* $\mathbf{f}$ *at* $\mathbf{a}$ *from the direction* $\mathbf{v}$ *exists and further*

$$\lim_{t\to 0+} \mathbf{f}(\mathbf{a}+t\mathbf{v}) = \mathbf{b}.$$

Prove this.

Hint This is a particular form of the Composite Rule for limits and so we can follow the outline of all proofs of such results.

Start by considering the ε - δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ and

finish by verifying the ε - δ definition of $\lim_{t\to 0+} \mathbf{f}(\mathbf{a}+t\mathbf{v}) = \mathbf{b}$.

The result of the previous question can be written symbolically as

$$\boxed{\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \implies \forall\, \mathbf{v},\ \lim_{t\to 0+} \mathbf{f}(\mathbf{a} + t\mathbf{v}) = \mathbf{b}.} \qquad (1)$$

The contrapositive can be used to prove limits do not exist.

8. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{for } \mathbf{x} = (x, y)^T \neq \mathbf{0} \quad \text{and} \quad f(\mathbf{0}) = 0.$$

i. Find $\lim_{t\to 0+} f(t\mathbf{e}_1)$ and $\lim_{t\to 0+} f(t\mathbf{e}_2)$ where $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$ are the two standard basis vectors for $\mathbb{R}^2$.

ii. Prove that f does not have a limit at $\mathbf{0}$.

9. **Lemma from lecture notes, limits along curves**. *Assume* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *where* A *contains a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$. *Assume* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ *exists. Assume* $\mathbf{g} : (0, \eta) \to A \setminus \{\mathbf{a}\}$ *with* $\lim_{t\to 0+} \mathbf{g}(t) = \mathbf{a}$. *Then*

$$\lim_{t\to 0+} \mathbf{f}(\mathbf{g}(t)) = \mathbf{b}.$$

Prove this.

Hint Question 7 is a special case of this result, so use the same method of proof. **Start** by looking at the ε - δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ and **finish** verifying the ε - δ definition of $\lim_{t\to 0+} \mathbf{f}(\mathbf{g}(t))$.

10. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{(x^2 - y)^2}{x^4 + y^2} \quad \text{for } \mathbf{x} = (x, y)^T \neq \mathbf{0} \quad \text{and} \quad f(\mathbf{0}) = 1.$$

i. Prove that $\lim_{t\to 0+} f(t\mathbf{v}) = 1$ for every non-zero vector $\mathbf{v}$.

   Hint: write $\mathbf{v} = (h, k)^T$ in order to get an expression for $f(t\mathbf{v})$. Be careful when $k = 0$.

ii. By considering the limit along the curve that is the image of $\mathbf{g}(t) = (t, t^2)^T$, prove that the function f does **not** have a limit at 0.

Hint Use the result of Question 9.

This is an example promised in the notes, of a function where the directional limit exists and are equal for all directions but the limit does not exist. That is

$$\boxed{\forall \mathbf{v},\ \lim_{t\to 0+} \mathbf{f}(\mathbf{a} + t\mathbf{v}) = \mathbf{b} \ \not\Rightarrow\ \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}.}$$

That is, the converse of (1) *is false.*

The following is a particularly important question. Make sure you attempt all parts which illustrate points made in the lectures and are used in later questions.

11. Find the following limits if they exist:

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$; (ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$;

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$; (iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$;

(v) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^2}$; (vi) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$.

Hint First try to show they have no limit by finding different directions (and even different curves) along which the function has different limits. If you cannot find any counterexamples try to prove the limit exists, normally by applying the Sandwich Rule.

For an example of the limit of a *vector*-valued function we have

12. Find the limit, if it exists, of

$$\lim_{(x,y)\to(0,0)} \left(x^2y+1, \frac{(xy)^2}{(xy)^2+(x-y)^2} \right)^T .$$

Additional Questions 1

13. By verifying the ε - δ definition show that the scalar-valued $g : \mathbb{R}^3 \to \mathbb{R}$, $(x, y)^T \mapsto x^2 y$ has limit 12 at $\mathbf{a} = (2, 3)^T$.

Hint Prove that

$x^2 y - 12 = (x-2)^2 (y-3) + 3(x-2)^2 + 4(x-2)(y-3) + 12(x-2) + 4(y-3)$.

Deduce that if $|x-2|$, $|y-3| < \delta$ and $\delta \leq 1$ then

$$\left|x^2 y - 12\right| < 24\delta.$$

14 Verify that the vector-valued function

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x^2 y \end{pmatrix}$$

has limit $(5, 12)^T$ at $\mathbf{a} = (2, 3)^T$.

Note You are not required to verify the definition.

15. *In the lectures we need to use the Cauchy-Schwarz inequality* $|\mathbf{a} \bullet \mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$ *and the triangle inequality* $|\mathbf{c} + \mathbf{d}| \leq |\mathbf{c}| + |\mathbf{d}|$, *for vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$. *This question is a recap of proofs of these results which you should already know.*

i. Prove that if $a, b, c \in \mathbb{R}$, $a > 0$ and $ax^2 + 2bx + c \geq 0$ for all $x \in \mathbb{R}$ then $b^2 \leq ac$. When do we have equality?

ii. Starting from the true statement that

$$0 \leq \sum_{i=1}^{n} (a_i + b_i x)^2$$

for all $x \in \mathbb{R}$, deduce the Cauchy-Schwarz inequality $|\mathbf{a} \bullet \mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$, written in the form

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

When do we have equality?

Hint *Make use of Part i.*

iii. *Triangle inequality.* Prove that if $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ then $|\mathbf{c} + \mathbf{d}| \leq |\mathbf{c}| + |\mathbf{d}|$.

Hint: *make use of part iii.*

iv. Prove that if $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ then $|\mathbf{c} - \mathbf{d}| \geq ||\mathbf{c}| - |\mathbf{d}||$.